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SEMIRIGID SPACES

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ABSTRACT. Semirigid spaces are introduced and used as a means to construct two metrizable spaces with isomorphic monoids of continuous self-maps and nonisomorphic clones; this resolves Problem 1 in [13]. The clone of any free variety of a given type with sufficiently many constants is shown to be isomorphic to the clone of a metrizable (semirigid) space.

I. INTRODUCTION

In June 1991, quoting a lecture by W. Taylor, J. R. Isbell turned my attention to the question of existence of two topological spaces (as nice as possible) whose monoids of continuous self-maps are isomorphic while their clones are not. (We recall that *the clone* $\mathrm{Clo}(X)$ *of a topological space* X is the full subcategory of Top, the category of all topological spaces and continuous maps, generated by all finite powers X^n of the space X.)

Since the monoid of all continuous self-maps of a space is usually rather large and hence somewhat difficult to handle, it is natural to try to make it as small as possible. The extreme case is that of a *rigid* space, that is, any space X whose all nonidentity continuous self-maps are constant (the first rigid T_1-space X with $\mathrm{card}\, X > 1$ was constructed by de Groot in [5]). Any two nonhomeomorphic rigid spaces of equal cardinality have isomorphic monoids of all continuous self-maps because these monoids consist of the identity and the same number of left zeros. But they also have isomorphic clones! The latter fact follows immediately from the well-known result that every continuous map $f : X^n \to X$ of a rigid space X with $\mathrm{card}\, X \geq 3$ is either constant or a projection (for Hausdorff spaces and n arbitrary, see [6, 7]; for the above formulation, see [13]).

In the present paper, we relax the rigidity and introduce the notion of semirigid spaces. While this notion will exert partial control over the monoid of all continuous self-maps, it is not strict enough to eliminate all nonconstant maps and thus force the clones to be isomorphic. We construct two metrizable (semirigid) spaces with the same monoid of continuous self-maps and nonisomorphic clones. In fact, more general results are presented here.

Since semirigid spaces may well be of interest in themselves and could become of use for problems other than those investigated here, §§II, III, IV, and the proof of the Main Theorem in §VII are written for topologists not involved in universal algebra: the definition of the clone of a space, as presented above,

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is all that is needed for these sections. On the other hand, §§V and VI use some notions and methods of universal algebra (e.g., the first-order logic for many-sorted algebras) and the proofs are often fairly brief. A more detailed description of the paper's contents is presented in §II.4.

II. Semirigid spaces and the main theorem

II.1. Let us begin with a definition.

Definition. A point x of a topological space X is called *rigid* if every continuous self-map $f: X \to X$ with $x \in f(X)$ is either identity or constant. A space X is called *semirigid* if it has a rigid point.

Remark. Clearly, every semirigid space is connected. Let us denote by B a subset of a space X such that every $x \in X \setminus B$ is a rigid point of X. Then every continuous $f: X \to X$ is either identity or constant or $f(X) \subseteq B$. Then we say that X is *B-semirigid.*

II.2. **Definition.** We say that a topological space $X = (P, t)$ is *absolutely B-semirigid* if every (P, t') is B-semirigid whenever t' is a Hausdorff topology coarser than t, B is t'-closed and t' coincides with t on $P \setminus B$ (i.e., $t'|P \setminus B = t|P \setminus B$). If, moreover, $t|B$ is discrete, we say that X is *extremally B-semirigid* (and, if $P \setminus B \neq \varnothing$, we say that X is *extremally semirigid*).

Remark. Since every semirigid space is connected, every extremally semirigid space is necessarily rigid.

II.3. Our goal is the following

Main Theorem. *For any sets P and $B \subseteq P$ such that*

$$\operatorname{card}(P \setminus B) = \operatorname{card} B \geq 2^{\aleph_0}$$

there exists a metrizable topology t on P such that $X = (P, t)$ is extremally B-semirigid.

Since the proof of the Main Theorem is rather involved and its method and constructions differ considerably from other constructions presented in this paper (they are more relevant to the constructions in [14] and [15]), we postpone it until §VII. In §§II–VI, we use only the fact that, for every cardinal $\alpha \geq 2^{\aleph_0}$, there exists an extremally B-semirigid metrizable space (P, t) with $\operatorname{card} B = \operatorname{card}(P \setminus B) = \alpha$. Its internal structure is quite irrelevant.

II.4. Let us describe the contents of the paper. Given an extremally B-semirigid space $X = (P, t)$, we can impose an arbitrary Hausdorff topology u on B and extend it in a suitable way so that the resulting space is still semirigid. This is used in §III, where semirigid metric spaces $X = (P, \tau)$ and $Y = (P, \tau')$ are constructed by an iterative procedure. In §IV,we prove that X and Y have the same monoid of all continuous selfmaps (by showing that these maps are in one-to-one correspondence to finite binary trees with leaves labeled by elements of $P \setminus B$ and by a symbol $\mathbb{I}$), and that the full subcategory of Top generated by X and X^2 admits no full embedding into the full subcategory of Top generated by the powers of Y. In §V, generalizing §§III and IV, we present a first order sentence T_n of the clone theory (a parametric sentence depending on n) such that for every $n > 1$ there exist metric spaces X_n and

Y_n whose clones have the same initial segment up to $n-1$ and T_n is satisfied in $\mathrm{Clo}(X_n)$ but not in $\mathrm{Clo}(Y_n)$. This solves Problem 1 in [13]. Which clones can be represented as clones of topological spaces? Section VI shows that the clone of every free variety with enough constants is isomorphic to the clone of a metrizable space. Finally, §VII demonstrates the Main Theorem (and its strengthening, in §VII.14).

II.5. We conclude this section by a proposition analogous to a Herrlich's result on rigid spaces.

Proposition. *Let X be a B-semirigid space with* $\mathrm{card}(X \setminus B) \geq 3$ *and let $g\colon X^n \to X$ be a continuous map. If n is finite or X is Hausdorff, then either g is constant or g is a projection or $g(X^n) \subseteq B$.*

Proof. Let $X^n = \prod_{i\in n} X_i$, $X_i = X$; for every $i \in n$, we denote by $\pi_i\colon X^n \to X$ the ith projection. Given $a \in X^n$ and $i \in n$, we denote by $e_{i,a}: X \to X^n$ the embedding sending any $x \in X$ to a point which differs from a in at most the ith coordinate and its ith coordinate is x.

Let a continuous map $g\colon X^n \to X$ be given. We investigate the following cases.

(1) There exist $a \in X^n$ and $i \in n$ such that $g \circ e_{i,a}$ is equal to the identity map $1_X : X \to X$.

We prove that, in this case, $g = \pi_i$.

(1,a) Choose $j \in n$, $j \neq i$. Choose $b \in X^n$ such that $\pi_i(b) \notin B \cup \{\pi_j(a)\}$ and $\pi_l(b) = \pi_l(a)$ for all $l \in n \setminus \{i\}$. Then $g \circ e_{j,b}$ is constant with the value $\pi_i(b)$. In fact,

$$e_{j,b}(\pi_j(b)) = b = e_{i,b}(\pi_i(b)),$$

so that

$$[g \circ e_{j,b}](\pi_j(a)) = [g \circ e_{j,b}](\pi_j(b)) = g(b) = [g \circ e_{i,b}](\pi_i(b)) = \pi_i(b).$$

Since $\pi_j(a) \neq \pi_i(b)$, $g \circ e_{j,b}$ is not the identity. Since $\pi_i(b) \notin B$, $g \circ e_{j,b}$ cannot map the whole X^n into B. Hence $g \circ e_{j,b}$ must be constant with the the value $\pi_i(b)$.

(1,b) Choose $j \in n$, $j \neq i$. We show that

> for every $z \in X^n$ which differs from a in at most the jth coordinate, we have $g \circ e_{i,z} = 1_X$.

In fact, choose b, c in X^n such that $\pi_l(b) = \pi_l(c) = \pi_l(a)$ for all $l \in n \setminus \{i\}$ and $\pi_i(b)$, $\pi_i(c)$ are two distinct elements of $X \setminus (B \cup \pi_j(a))$. By (1,a), $g \circ e_{j,b}$ is constant with the value $\pi_i(b)$ and $g \circ e_{j,c}$ is constant with the value $\pi_j(c)$. Denote by b' (resp. c') the point of X^n which differs from b (resp. c) in at most the jth coordinate and this jth coordinate is equal to $\pi_j(z)$. Then

$$e_{i,z}(\pi_i(b)) = b' = e_{j,b}(\pi_j(z)) \quad \text{and} \quad e_{i,z}(\pi_i(c)) = c' = e_{j,c}(\pi_j(z)).$$

Hence

$$[g \circ e_{i,z}](\pi_i(b)) = [g \circ e_{j,b}](\pi_j(z)) = \pi_i(b)$$

and

$$[g \circ e_{i,z}](\pi_i(c)) = [g \circ e_{j,c}](\pi_j(z)) = \pi_i(c).$$

Since $\pi_i(b) \neq \pi_i(c)$, $g \circ e_{i,z}$ cannot be constant. Since $\pi_i(b) \notin B$, $g \circ e_{i,z}$ cannot map X into B. Hence $g \circ e_{i,z}$ must be the identity.

(1,c) If z differs from a in at most one coordinate, then $g \circ e_{i,z} = 1_X$, by (1,b). If y differs from z in at most one coordinate, then $g \circ e_{i,y} = 1_X$, by (1,b) again. Now we proceed by induction; finally, we conclude that, if z differs from a in finitely many coordinates, then $g \circ e_{i,z} = 1_X$. This already means that $g = \pi_i$ whenever n is finite (so that every $y \in X^n$ differs in at most finitely many coordinates) or whenever X is Hausdorff (hence $g : X^n \to X$ is uniquely determined by its values on the dense subspace of X^n).

(2) The statement in (1) is not valid, i.e., $g \circ e_{i,a}$ is never equal to 1_X. We subdivide this case as follows.

(2,1) There exist $i \in n$, $a \in X^n$, and $d \in X$ such that $[g \circ e_{i,a}](d) \notin B$. We show that g is constant.

(2,1,a) Since $g \circ e_{i,a}$ is not 1_X and it maps $d \in X$ in $X \setminus B$, it must be constant. Denote by b this unique value of $g \circ e_{i,a}$.

(2,1,b) Choose $j \in n$, $j \neq i$. Since $g \circ e_{j,a}$ is not 1_X, it is either constant or it maps the whole X into B. The latter case is impossible because $[g \circ e_{j,a}](\pi_j(a)) = [g \circ e_{i,a}](\pi_i(a)) = b \notin B$.

(2,1,c) By (2,1,b), g is constant with the value b on the set of all $z \in X^n$ which differ from a in at most one coordinate. We proceed by induction and conclude, as in (1,c), that g is constant whenever n is finite or X is Hausdorff.

(2,2) The statement (2,1) is not valid. In this case it is clear that g maps the whole X^n into B.

III. Iterative constructions

III.1. We wish the constructed topologies to be metrizable. Hence we shall deal with metrics rather than topologies. Some (well-known) lemmas about pseudometrics will be of use.

Standing convention. Throughout the paper, "pseudometric" always means the pseudometric

less than or equal to 1;

we shall not repeat it explicitly. If (P, ϱ) is a pseudometric space, then $\varrho \times \varrho$ is the pseudometric on $P \times P$ given by

$$\varrho \times \varrho((x_1, x_2), (y_1, y_2)) = \max(\varrho(x_1, y_1), \varrho(x_2, y_2)).$$

III.2. *Observation.* Let (P, t) be a metrizable space, B its closed discrete subset. Then (P, t) can be metrized by a metric ϱ such that B is 1-*discrete* (i.e., $x, y \in B$, $x \neq y \Rightarrow \varrho(x, y) = 1$).

[In fact, we set $\varrho = \min(\tilde{\varrho}, 1)$, where

$$\tilde{\varrho}(x, y) = \sigma(x, y) + \sum_{b \in B} |f_b(x) - f_b(y)| ,$$

σ metrizes (P, t), and $\{f_b \mid b \in B\}$ is a locally finite system of real continuous functions with $0 \leq f_b \leq 1$ and $f_b(b) = 1$, $f_b(a) = 0$ for all $a \in B$, $a \neq b$.]

III.3. *Observation.* Let a metric space (P, ϱ) be given, and let B be its closed subset. Let u be a pseudometric on B, and let

$$u(a, b) \leq \varrho(a, b) \quad \text{for all } a, b \in B.$$

For every $x, y \in P$, set

$$\sigma(x, y) = \min\left(\varrho(x, y), \inf_{a, b \in B}(\varrho(x, a) + u(a, b) + \varrho(b, y))\right).$$

Then σ is a pseudometric on P and

(a) $\sigma \le \varrho$ and $\sigma(a, b) = u(a, b)$ for all $a, b \in B$;
(b) for every $x, y \in P$, $\sigma(x, y) \ge \min(\varrho(x, y), \varrho(x, B) + \varrho(y, B))$;
(c) for every $x \in P \setminus B$ and $y \in B$, $\sigma(x, y) \ge \varrho(x, B)$;
(d) $P \setminus B$ is open in (P, σ) and both ϱ and σ determine the same topology on it.

Notation. Let us denote the pseudometric σ by $\varrho * u$.

III.4. *Observation.* Let (P, ϱ) and B be as in III.3. Let u_1, u_2 be pseudometrics on B such that

$$u_2(a, b) \le u_1(a, b) \le \varrho(a, b) \quad \text{for all } a, b \in B.$$

Then $\varrho * u_2 \le \varrho * u_1$.

III.5. *Standing hypothesis.* Let P, $B \subseteq P$ be sets, $\operatorname{card} B = \operatorname{card} P \setminus B \ge 2^{\aleph_0}$. Let $B = \bigcup_{n=1}^{\infty} B_n$, $B_i \cap B_j = \varnothing$ for $i \ne j$, $\operatorname{card} B_i = \operatorname{card} B$. Set $G_0 = P \setminus B$, $G_k = G_0 \cup \bigcup_{i=1}^{k} B_i$. Let $g : P \times P \to B$ be a bijection of $P \times P$ onto B such that g maps $G_0 \times G_0$ onto B_1 and g maps $(G_n \times G_n) \setminus (G_{n-1} \times G_{n-1})$ onto B_{n+1} for all $n \ge 1$.

III.6. *Construction.* Let (P, ϱ) be a metric space, B its 1-discrete subset. Let $B = \bigcup_{n=1}^{\infty} B_n$, G_k, g be as in III.5. We define a descending chain of pseudometrics τ_α on P $(\alpha \in \mathrm{Ord})$ by transfinite induction as follows:

$\alpha = 0 : \tau_\alpha = \varrho$,
α limit ordinal : $\tau_\alpha = \inf_{\beta < \alpha} \tau_\beta$,
α isolated, $\alpha = \beta + 1 : \tau_\alpha = \varrho * u_\beta$, where u_β is the pseudometric on B for which g is an isometry of $(P \times P, \tau_\beta \times \tau_\beta)$ onto (B, u_β).

Since ϱ is 1-discrete on B, we have $u_\beta \le \varrho | B \times B$ so that the definition of τ_α is correct. By III.4, we obtain a descending chain of pseudometrics.

Since there is only a set of distinct pseudometrics on P, this procedure must stop; hence there exists an ordinal α such that $\tau_\alpha = \tau_{\alpha+1} = \tau_{\alpha+2} = \cdots$. Let τ denote the resulting pseudometric. Clearly,

$$g \text{ is an isometry of } (P \times P, \tau \times \tau) \text{ onto } (B, \tau | B \times B).$$

III.7. **Proposition.** *The pseudometric τ is a metric.*

Proof. We construct a real symmetric function f on $P \times P$ such that $0 \le f \le 1$, $f(x, x) = 0$ and $f(x, y) > 0$ for all $x, y \in P$, $x \ne y$, and we show that $\tau_\alpha(x, y) \ge f(x, y)$ for all $x, y \in P$ and all ordinals α.

We construct f inductively as follows:

(1) for $x, y \in G_0$, we set $f(x, y) = \min(\varrho(x, y), \varrho(x, B) + \varrho(y, B))$;
(2) if f is defined for all $x, y \in G_n$, we extend the definition on G_{n+1}:
 (a) for $x, y \in G_{n+1} \setminus G_n$, we define

$$f(x, y) = \max(f(x_1, y_1), f(x_2, y_2)),$$

where $(x_1, x_2) = g^{-1}(x)$ and $(y_1, y_2) = g^{-1}(y)$;

(b) for $x \in G_{n+1} \setminus G_n$ and $y \in G_n \cap B$, we define

$$f(x, y) = f(y, x) \text{ as in (a);}$$

(c) for $x \in G_{n+1} \setminus G_n$ and $y \in G_0$, we define

$$f(x, y) = f(y, x) = \varrho(y, B).$$

One can see easily that $f(x, y) > 0$ whenever $x \neq y$ and

(i) $f(x, y) = \varrho(y, B)$ if $y \in G_0$ and $x \in B$;
(ii) $f(x, y) = \max(f(x_1, y_1), f(x_2, y_2))$ for all $x, y \in B$, where $(x_1, x_2) = g^{-1}(x)$ and $(y_1, y_2) = g^{-1}(y)$.

By (i) and (1) above, we get $\varrho \geq f$. Hence $\tau_\alpha \geq f$, by transfinite induction, using (ii) and III.3–6.

III.8. **Proposition.** *Let* τ *be as in* III.6–7. *Then* G_0 *is an open subset of the metric space* (P, τ), *and the metrics* ϱ *and* τ *determine the same topology on* G_0.

Proof. For $x, y \in G_0$, we have

$$\varrho(x, y) \geq \tau(x, y) \geq \min(\varrho(x, y), \varrho(x, B) + \varrho(y, B));$$

for $x \in G_0$ and $y \in B$, we have $\varrho(x, y) \geq \tau(x, y) \geq \varrho(x, B)$.

III.9. *Generalization of the construction in* III.6. Let (P, ϱ), $B = \bigcup_{n=1}^{\infty} B_n$, G_k, g be as in III.5–6. Moreover, let $\mathbb{R}$ be an abstract rule that, for every pseudometric σ on P, gives a pseudometric $\mathbb{R}(\sigma)$ on $P \times P$ such that

(α) $\sigma \times \sigma \leq \mathbb{R}(\sigma)$ and
(β) $\mathbb{R}$ is monotone, i.e., if σ, μ are pseudometrics on P such that $\sigma \leq \mu$, then $\mathbb{R}(\sigma) \leq \mathbb{R}(\mu)$.

Following the construction in III.6, we set

$$\tau'_0 = \varrho,$$
$$\tau'_\alpha = \inf_{\beta<\alpha} \tau'_\beta \text{ for any limit ordinal } \alpha,$$
$$\tau'_\alpha = \varrho * u'_\beta \text{ for } \alpha = \beta + 1, \text{ where } u'_\beta \text{ is the pseudometric on } B \text{ such that } g \text{ is an isometry of } (P \times P, \mathbb{R}(\tau'_\beta)) \text{ onto } (B, u'_\beta).$$

This procedure must stop again. Let τ' denote the resulting pseudometric.

Since, for every α, $\tau_\alpha \times \tau_\alpha \leq \mathbb{R}(\tau_\alpha)$, by transfinite induction we obtain $\tau_\alpha \leq \tau'_\alpha$ for all α, and hence $\tau \leq \tau'$. Consequently, τ' is a metric. We also have $\varrho \geq \tau'$ and

$$\tau'(x, y) \geq \varrho(x, B) \quad \text{for } x \in G_0,\ y \in B,$$
$$\tau'(x, y) \geq \min(\varrho(x, y), \varrho(x, B) + \varrho(y, B)) \quad \text{for } x, y \in G_0.$$

We conclude that (P, τ') is a metric space such that

G_0 is open in (P, τ'), ϱ and τ' determine the same topology on it, and g is an isometry of $(P \times P, \mathbb{R}(\tau'))$ onto $(B, \tau'|B \times B)$.

We will use this in IV.5 for a specific rule $\mathbb{R}$.

IV. Maps and trees

IV.1. Let (P, ϱ), $B = \bigcup_{n=1}^{\infty} B_n$, G_k, g be as in III.5–6, and let τ and τ' be as in III.6 and III.9. Now, we add the fact that

(P, ϱ) is extremally B-semirigid.

This can be assumed, by the Main Theorem. (We can metrize P by a metric ϱ such that B is 1-discrete, by III.2. In VII, however, this space will be constructed directly with such a metric ϱ.)

Here, we describe continuous selfmaps of (P, τ) by suitably labeled finite binary trees. To do this, we begin with a brief presentation of a standard tree-handling technique, in a form suitable for our purpose (for a general approach, see, e.g., [1]).

IV.2. A *finite binary tree* is a finite nonvoid set T of words of elements 0, 1, r (=root) such that

(1) if $w = w_0 \cdots w_n \in T$, then $w_0 = r$ and $w_1, \dots, w_n \in \{0, 1\}$;
(2) if $rw_1 \cdots w_n \in T$, then
  (a) all words r, rw_1, rw_1w_2, $\dots$, $rw_1 \cdots w_{n-1}$ are also in T;
  (b) $rw_1 \cdots w_{n-1}\tilde{w}_n \in T$, where $\{w_n, \tilde{w}_n\} = \{0, 1\}$.

Maximal words of T are its *leaves*, all other words in T are its *nodes*. [Hence, for instance, r is a leaf of T iff $T = \{r\}$; $T = \{r, r0, r1\}$ has precisely one node and two leaves.] Define a binary operation $\circ$ on the set of all finite binary trees by

$$T \circ T' = \{r\} \cup \{r0w_1 \cdots w_n | rw_1 \cdots w_n \in T\} \cup \{r1w_1 \cdots w_n | rw_1 \cdots w_n \in T'\}.$$

The set of all leaves of $T \circ T'$ is in an obvious one-to-one correspondence with the disjoint union of the set of all leaves of T and of T'.

Given a set A, an *A-labeled tree* is a pair (T, l) where T is a finite binary tree and l is a map of the set of all its leaves into A. If (T_1, l_1) and (T_2, l_2) are A-labeled trees, then $(T_1, l_1) \circ (T_2, l_2)$ is the A-labeled tree (T, l) for which $T = T_1 \circ T_2$ and

$$l(r0w_1 \cdots w_n) = l_1(rw_1 \cdots w_n),$$
$$l(r1w_1 \cdots w_n) = l_2(rw_1 \cdots w_n).$$

Clearly, every (T, l) with $T \neq \{r\}$ can be expressed uniquely as $(T, l) = (T_1, l_1) \circ (T_2, l_2)$.

IV.3. Let P, $B = \bigcup_{n=1}^{\infty} B_n$, $G_0 = P \setminus B$, G_k, $g \colon P \times P \to B$ be as in III.5. Let $A = G_0 \cup \{\mathbb{I}\}$, where $\mathbb{I}$ is an abstract element not in G_0. We describe an algorithm $\mathscr{A}$ which sends every A-labeled tree (T, l) to a self-map $\mathscr{A}(T, l) \colon P \to P$.

(i) $$\mathscr{A}(\{r\}, l) = \begin{cases} \text{the identity } P \to P \text{ whenever } l(r) = \mathbb{I}; \\ \text{the constant map with the value } x \text{ whenever } l(r) = x. \end{cases}$$

(ii) $\mathscr{A}(T, l) = g \circ [\mathscr{A}(T_1, l_1) \dot{\times} \mathscr{A}(T_2, l_2)]$ otherwise,

where the symbol $\alpha \dot{\times} \beta$ means the map $P \to P \times P$ sending every $p \in P$ to $(\alpha(p), \beta(p))$.

IV.4. Next, we add the metric τ constructed in III.6, that is, in addition to the standing hypothesis III.5 we assume that

$$(*) \qquad \begin{cases} (P, \tau) \text{ is a } B\text{-semirigid metric space and} \\ g \text{ is an isometry of } (P \times P, \tau \times \tau) \text{ onto } (B, \tau | B \times B). \end{cases}$$

In this case, we show that the above algorithm $\mathscr{A}$ gives a bijection of the set of all A-labeled trees onto the set of all continuous self-maps of (P, τ).

(α) First, we have to verify that if (T, l) is an A-labeled tree, then $\mathscr{A}(T, l) : (P, \tau) \to (P, \tau)$ is continuous. This follows immediately from (i) and (ii) in IV.3.

(β) Now, we describe a procedure $\mathscr{A}^{-1}$ inverse to $\mathscr{A}$: given a continuous map $f : (P, \tau) \to (P, \tau)$, we construct $(T, l) = \mathscr{A}^{-1}(f)$: Since (P, τ) is B-semirigid, either f is the identity (and then $\mathscr{A}^{-1}(f) = (\{r\}, l)$ with $l(r) = \mathbb{I}$) or a constant with a value $x \in G_0$ (and then $\mathscr{A}^{-1}(f) = (\{r\}, l)$ with $l(r) = x$) or $f(P) \subseteq B$. In the last case, we compose f with g^{-1} and with the first or the second projection, and investigate the two maps $\pi_1 \circ g^{-1} \circ f$, $\pi_2 \circ g^{-1} \circ f : P \to P$.

If both of these maps are identities, we get $\mathscr{A}^{-1}(f) = (\{r, r0, r1\}, l)$ with $l(r0) = \mathbb{I} = l(r1)$; if one of them is the identity and the other is constant with a value $x \in G_0$, we take the same tree $\{r, r0, r1\}$ and the corresponding labeling. If either $\pi_1 \circ g^{-1} \circ f$ or $\pi_2 \circ g^{-1} \circ f$ maps P into B, the procedure continues in the evident way. We only have to show that the procedure $\mathscr{A}^{-1}$ always stops after a finite number of steps.

For a continuous map $f : (P, \tau) \to (P, \tau)$ thus $\mathscr{A}^{-1}$ stops for f in one step or else $f(P) \subseteq B$. Let n be a natural number such that $f(P) \cap B_n \neq \varnothing$. We show that the procedure $\mathscr{A}^{-1}$ stops after at most $n+1$ steps. We prove it by induction. If $n = 1$, then $[g^{-1} \circ f](P)$ intersects $G_0 \times G_0$ so that both $[\pi_1 \circ g^{-1} \circ f](P)$ and $[\pi_2 \circ g^{-1} \circ f](P)$ intersect G_0. Hence both must be either the identity or constant with a value in G_0 and the procedure $\mathscr{A}^{-1}$ stops at this second step. If $n > 1$ and $f(P)$ intersects B_n, then $[g^{-1} \circ f](P)$ intersects $G_{n-1} \times G_{n-1}$, i.e., both $[\pi_1 \circ g^{-1} \circ f](P)$ and $[\pi_2 \circ g^{-1} \circ f](P)$ intersect $G_{n-1} = G_0 \cup \bigcup_{i=1}^{n-1} B_i$ so that, by the induction hypothesis, the procedure $\mathscr{A}^{-1}$ stops in at most n steps for both $\pi_1 \circ g^{-1} \circ f$ and $\pi_2 \circ g^{-1} \circ f$, and hence in at most $n+1$ steps for f.

The fact that the procedure $\mathscr{A}^{-1}$ is inverse to the procedure $\mathscr{A}$ is evident.

IV.5. Suppose that, in addition to the standing hypothesis III.5, we have an abstract rule $\mathbb{R}$ as in III.9. In IV.4, replace the additional data by the statement

$$(*') \qquad \begin{cases} (P, \tau') \text{ is a } B\text{-semirigid metric space and} \\ g \text{ is an isometry of } (P \times P, \mathbb{R}(\tau')) \text{ onto } (B, \tau'|B \times B). \end{cases}$$

Next we analyze conditions on the rule $\mathbb{R}$ under which the algorithm $\mathscr{A}$ of IV.3 gives a bijection between the set of all A-labeled trees and the set of all continuous self-maps of (P, τ'). In IV.4 (β), we needed only the continuity of the projections $\pi_1, \pi_2 : (P \times P, \mathbb{R}(\tau')) \to (P, \tau')$; the topology of $(P \times P, \mathbb{R}(\tau'))$ thus must be finer than the topology of $(P \times P, \tau' \times \tau')$. This is surely satisfied whenever $\tau' \times \tau' \leq \mathbb{R}(\tau')$—as required already in III.9—and $\mathscr{A}^{-1}$ forces no new restrictions. On the other hand, from IV.4 (α) it follows that we need that

$$(+) \qquad \begin{aligned} &\text{if } \alpha, \beta : (P, \tau') \to (P, \tau') \text{ are continuous, then} \\ &\alpha \dot{\times} \beta : (P\tau') \to (P \times P, \mathbb{R}(\tau')) \text{ is continuous.} \end{aligned}$$

This is a real restriction. However, because of the semirigidity, there are still many rules $\mathbb{R}$ that satisfy (+) and the topologies given by $\tau' \times \tau'$ and $\mathbb{R}(\tau')$ are distinct (and hence the topologies of (P, τ) and of (P, τ'), constructed in III.6 and III.9, are also distinct). One such rule $\mathbb{R}$ is described by IV.6.

IV.6. Let P, B, G_0, G_k, g be as in III.5, let ϱ be a metric on P such that (P, ϱ) is extremally B-semirigid and B is 1-discrete. Let a, b be two distinct points of G_0. We change the topology of $(P \times P, \varrho \times \varrho)$ at the point (a, b) as follows.

We choose $\varepsilon > 0$, $\varepsilon < \min(\varrho(a, B), \varrho(b, B), \varrho(a, b))$, and set

$$F_1 = \{(x, y) \in P \times P | \varrho(x, a) \geq \varepsilon \text{ or } \varrho(y, b) \geq \varepsilon\},$$
$$F_2 = \{(x, y) \in P \times P | 0 < \varrho(x, a) = \varrho(y, b) \leq \varepsilon\},$$
$$F_3 = \{(x, y) \in P \times P | x = a \text{ or } y = b\} \setminus \{(a, b)\}.$$

Then $F_1 \cup F_2 \cup F_3$ is a closed subset of the subspace $Q = P \times P \setminus \{(a, b)\}$ of $(P \times P, \varrho \times \varrho)$. Let f be a real continuous function on Q such that $0 \leq f \leq 1$ and

$$f(x, y) = 0 \quad \text{for all } (x, y) \in F_1 \cup F_3,$$
$$f(x, y) = \frac{1}{\varepsilon}(\varepsilon - \varrho(x, a)) \quad \text{for all } (x, y) \in F_2.$$

Finally, we set $f(a, b) = 0$. The function f is thus defined on $(P \times P, \varrho \times \varrho)$ and is continuous everywhere except at the point (a, b), where it is discontinuous: since (P, ϱ) is connected, for every $0 < \delta \leq \varepsilon$ there exist $x, y \in P$ with $\varrho(x, a) = \varrho(y, b) = \delta$; clearly, if (x, y) approaches to (a, b), then $\lim f(x, y) = 1$ when (x, y) is restricted to the points of F_2 and $\lim f(x, y) = 0$ for $(x, y) \in F_3$.

Given a pseudometric σ on P, we define the rule $\mathbb{R}$ by

$$[\mathbb{R}(\sigma)](x, y) = \min(1, (\sigma \times \sigma)(x, y) + |f(x) - f(y)|).$$

Now it is easy to see that if (P, τ') is the B-semirigid space constructed in III.9, then it fulfils $(+)$ in IV.5. In fact, if $\alpha, \beta : (P, \tau') \to (P, \tau')$ are continuous, then $\pi_1 \circ (\alpha \dot{\times} \beta)$, $\pi_2 \circ (\alpha \dot{\times} \beta) : (P, \tau') \to (P, \tau')$ are also continuous hence equal either to the identity or a constant, or map the whole P into B; in all possible cases, $\alpha \dot{\times} \beta : (P, \tau') \to (P \times P, \mathbb{R}(\tau'))$ is continuous because $\mathbb{R}(\tau')$ differs from $\tau' \times \tau'$ only in the neighbourhoods of the nondiagonal point $(a, b) \in G_0 \times G_0$, and the maps $x \rightsquigarrow (x, b)$ and $y \rightsquigarrow (a, y)$ are continuous.

One can also see easily that the maps

$$(\times) \qquad c, i : (P \times P, \tau' \times \tau') \to (P \times P, \mathbb{R}(\tau')) \text{ are not continuous,}$$

where $i(x, y) = (x, y)$ and $c(x, y) = (y, x)$ for all $(x, y) \in P \times P$.

IV.7. Let P, B, G_0, G_k, g be as in III.5, let ϱ be a metric on P such that (P, ϱ) is extremally B-semirigid and B is 1-discrete. Let τ be the metric on P constructed from these data as in III.6. Let $\mathbb{R}$ be the rule described in IV.6 and τ' be the metric on P constructed as in III.9. Then $f : P \to P$ is a continuous selfmap of $X = (P, \tau)$ iff it is a continuous self-map of $Y = (P, \tau')$. This follows from IV.1–6. Finally, we show that X and Y do not have isomorphic clones. In fact, we prove that

> the full subcategory k of Top generated by objects X and $X^2 = X \times X$ admits no full embedding into the full subcategory h of Top generated by *all* powers Y^α of Y $(\alpha \in \text{Card})$.

Proof. Let $\Phi : k \to h$ be full and faithful. Denote $Y^{\alpha_1} = \Phi(X)$, $Y^{\alpha_2} = \Phi(X^2)$.

(1) First we note that any constant morphism in $k(a, a)$, $a \in \operatorname{obj} k$, is sent to a constant morphism in $h(\Phi(a), \Phi(a))$. This follows from the fact that constants are precisely the left zeros of the endomorphism monoids in question.

(2) This implies that there are bijections $b_1 : P \to P^{\alpha_1}$, $b_2 : P \times P \to P^{\alpha_2}$ such that

$$b_j \circ f = \Phi(f) \circ b_i \quad \text{and} \quad b_j^{-1} \circ f' = \Phi^{-1}(f') \circ b_i^{-1}$$

whenever $\{i, j\} = \{1, 2\}$ and $f \in k(a_i, a_j)$, $f' \in h(\Phi(a_i), \Phi(a_j))$.

(3) The space $X = (P, \tau)$ is B-semirigid, hence every $x \in P \setminus B$ is rigid, so that $b_1(x)$ must be a rigid point of Y^{α_1}. This implies that $\alpha_1 = 1$ and $b_1 : P \to P$ sends $P \setminus B$ (and hence B) onto itself.

(4) Since there are precisely two distinct surjective morphisms $X^2 \to X$, the same is true for $Y^{\alpha_2} \to Y^{\alpha_1} = Y$, so that $\alpha_2 = 2$. The statement

(**) there exists a surjective morphism of $X \times X$ onto B

is fulfilled in k: the map $g : P\times P \to B$ has this property. Hence the analogous statement must be true for Y. We show that it is not true.

(5) Let $l: Y\times Y \to B$ be a continuous surjective map. Then $g^{-1}\circ l: Y\times Y \to (P\times P, \mathbb{R}(\tau'))$ is also a continuous surjection. Since $Y = (P, \tau')$ is B-semirigid, the maps $\pi_1 \circ g^{-1} \circ l$, $\pi_2 \circ g^{-1} \circ l$ are either projections or constants, or map the whole $P \times P$ into B, by II.5. Consequently $g^{-1} \circ l: (P \times P, \tau' \times \tau') \to (P \times P, \mathbb{R}(\tau'))$ is either i or c. But neither of these maps is continuous, see IV.6 ($\times$).

V. Clones of topological spaces

V.1. We have already defined the clone $\mathrm{Clo}(X)$ of a topological space X as the full subcategory of Top generated by all the finite powers of X. This is one way how to describe it. Another (equivalent) approach is to view it as an ω-sorted algebra (many sorted algebras = heterogeneous algebras, see [2])

$$\mathrm{Clo}(X) = (\{C_n\}_{n\in\omega}, \{S_n^m | n, m \in \omega\} \cup \{\pi_i^{(j)} | j \in \omega, 1 \le i \le j\})$$

where the underlying set of its nth sort C_n is the set of all continuous maps $X^n \to X$, each S_n^m is the operation sending the set $C_m \times C_n \times \cdots \times C_n$ (C_n appears here m times) into C_n by the rule

$$S_n^m(f, h_1, \ldots, h_m) = f \circ (h_1 \dot{\times} \cdots \dot{\times} h_m),$$

(where, for every $z \in X^n$, $[h_1 \dot{\times} \cdots \dot{\times} h_m](z) = (h_1(z), \ldots, h_m(z)) \in X^m$) and the projections $\pi_i^{(j)} : X^j \to X$, $i = 1, \ldots, j$, are viewed as constants in C_j (= nullary operations).

This description allows the use of all common notions introduced for many-sorted algebras, such as isomorphisms, homomorphisms, etc. In particular, it allows the introduction of first-order language of the many-sorted algebras and investigations of the first-order sentences about the clone. It also provides deep interrelations with other structures, see [13].

An *initial n-segment* of $\mathrm{Clo}(X)$ is its restriction to the sorts $C_0, \ldots, C_n$, to the operations S_i^k with $i, k \le n$, and to the constants $\pi_i^{(j)}$ with $1 \le i \le j \le n$; let us denote this $(n+1)$-sorted algebra by $\mathrm{Clo}_n(X)$.

V.2. In [13], the following problem is given: do there exist spaces X and Y such that $\mathrm{Clo}_1(X)$ and $\mathrm{Clo}_1(Y)$ satisfy exactly the same sentences of monoid theory but $\mathrm{Clo}(X)$ and $\mathrm{Clo}(Y)$ do not satisfy the same sentences of clone theory?

Our spaces $X = (P, \tau)$ and $Y = (P, \tau')$ have this property. However, much more can be obtained from suitable modifications of our construction. We shall present a first order sentence T_n of the initial n-segment of clone theory such that

> for every natural number $n > 1$ there exist metrizable spaces X, Y such that $\mathrm{Clo}_i(X)$ and $\mathrm{Clo}_i(Y)$ satisfy exactly the same sentences of the initial i-segment of clone theory for $i = 1, \dots, n-1$ and T_n is satisfied in $\mathrm{Clo}_n(X)$ but not satisfied in $\mathrm{Clo}_n(Y)$.

This also shows that first-order sentences about the whole clone have more expressive power than those about any of its initial segments.

V.3. Informally, the sentence T_n says that there exists a continuous map $g: X^n \to X$ such that $g(X^n)$ is precisely the set of all nonrigid points of X. Formally, it can be expressed as follows (where $x^{(i)}, y^{(i)}, \dots$ denote the variables of the ith sort and $\pi_j^{(i)}$, $j = 1, \dots, i$, denote the constants of the ith sort):

$$C(x^{(1)}) \equiv \forall y^{(1)}(S_1^1(x^{(1)}, y^{(1)}) = x^{(1)}),$$
$$x^{(1)} \in x^{(n)} \equiv C(x^{(1)}) \wedge \exists y_1^{(1)} \cdots \exists y_n^{(1)}(S_1^n(x^{(n)}, y_1^{(1)}, \dots, y_n^{(1)}) = x^{(1)}),$$
$$r(x^{(1)}) \equiv C(x^{(1)}) \wedge \forall y^{(1)}((x^{(1)} \in y^{(1)}) \Rightarrow (C(y^{(1)}) \vee (y^{(1)} = \pi_1^{(1)}))),$$
$$T_n \equiv \exists x^{(n)} \forall x^{(1)}(C(x^{(1)}) \Rightarrow (r(x^{(1)}) \Leftrightarrow \neg(x^{(1)} \in x^{(n)}))).$$

For a given natural number $n > 1$, the construction of the spaces $X = (P, \tau)$ and $Y = (P, \tau')$ such that $\mathrm{Clo}_i(X)$ and $\mathrm{Clo}_i(Y)$ satisfy the same sentences for $i = 0, 1, \dots, n-1$ and T_n is valid in $\mathrm{Clo}_n(X)$ but not in $\mathrm{Clo}_n(Y)$ is sketched below. It is a simple modification of the construction given in §§III and IV.

V.4. Let (P, ϱ), $B = \bigcup_{i=1}^{\infty} B_i$, $G_0 = P \setminus B$, $G_k = G_0 \cup \bigcup_{i=1}^{k} B_i$ be as in III.5–6. Let g be a bijection of P^n onto B such that g maps

$$G_0^n \text{ onto } B_1 \text{ and } G_i^n \setminus G_{i-1}^n \text{ onto } B_{i+1} \quad \text{for } i = 1, 2, \dots.$$

We construct the metric τ precisely as in III.6 (where we replace $(P \times P, \tau_\beta \times \tau_\beta)$ by (P^n, τ_β^n)), and prove that τ really is a metric analogously to III.7. Proposition III.8 is also valid. In IV.1–IV.4, we replace binary trees by n-ary trees and we get, analogously to IV, that continuous self-maps of $X = (P, \tau)$ are in one-to-one correspondence to all finite n-ary trees with leaves labeled by $\mathbb{I}$ and by elements of G_0 (this correspondence is given by the evident straightforward modification of the algorithm $\mathscr{A}$ in IV.3). The next immediate generalization, using II.5, gives us that

> for every $i = 1, 2, \dots$, the set of all continuous maps $X^i \to X$ is in one-to-one correspondence with the set of all finite n-ary trees with leaves labeled by the points of G_0 or the projections $\pi_1^{(i)}, \dots, \pi_i^{(i)}$: $X^i \to X$.

Procedures generalizing $\mathscr{A}$ and $\mathscr{A}^{-1}$ in IV.3–IV.4 are evident.

V.5. The space $Y = (P, \tau')$ is constructed as in III.9, using the map $g: P^n \to B$ and a monotone abstract rule $\mathbb{R}$ which, for every pseudometric σ on P, gives us a pseudometric $\mathbb{R}(\sigma)$ on P^n such that $\sigma^n \leq \mathbb{R}(\sigma)$. The rule $\mathbb{R}$ is then specified analogously to IV.6: we choose n distinct points $a_1, \ldots, a_n$ in G_0, $\varepsilon > 0$ with $\varepsilon < \min\{\min_{i \neq j, i, j=1, \ldots, n} \varrho(a_i, a_j), \min_{i=1, \ldots, n} \varrho(a_i, B)\}$ and set (with $a = (a_1, \ldots, a_n)$, $x = (x_1, \ldots, x_n)$)

$$\begin{aligned} F_1 &= \{x \in P^n | \varrho^n(x, a) \geq \varepsilon\}, \\ F_2 &= \{x \in P^n | 0 < \varrho(x_1, a_1) = \varrho(x_2, a_2) = \cdots = \varrho(x_n, a_n) \leq \varepsilon\}, \\ F_3 &= \{x \in P^n | (x_1, \ldots, \hat{x}_j, \ldots, x_n) = (a_1, \ldots, \hat{a}_j, \ldots, a_n), \\ &\qquad\qquad\qquad\qquad\qquad\qquad j = 1, \ldots, n\} \setminus \{a\}, \end{aligned}$$

where $\hat{x}_j$ and $\hat{a}_j$ denote the fact that the jth coordinate is skipped. Then we find a continuous real function f on $Q = P^n \setminus \{a\}$, $0 \leq f \leq 1$, such that

$$\begin{aligned} f(x) &= 0 \quad \text{for all } x \in F_1 \cup F_3, \\ f(x) &= (1/\varepsilon)(\varepsilon - \varrho(x_1, a_1)) \quad \text{for all } x \in F_2 \end{aligned}$$

and set $f(a) = 0$. Finally, we define

$$[\mathbb{R}(\sigma)](x, y) = \min(1, \sigma^n(x, y) + |f(x) - f(y)|).$$

Then we prove that the continuous maps $Y^k \to Y$ are in one-to-one correspondence with all n-ary trees whose leaves are labeled by the points of G_0 and the projections $\pi_j^{(k)}$, $j = 1, \ldots, k$, for $k = 1, \ldots, n-1$, that the bijection arises from the same algorithm as that for $X = (P, \tau)$, and that, for every permutation p of the set $\{1, \ldots, n\}$, the map $c_p : (P^n, (\tau')^n) \to (P^n, \mathbb{R}(\tau'))$, sending $(x_1, \ldots, x_n)$ to $(x_{p(1)}, \ldots, x_{p(n)})$, is not continuous. Finally, one can prove that Y does not satisfy the sentence T_n analogously as in IV.7 (where $(**)$ in IV.7 is in fact the sentence T_2).

VI. Topologies in free algebras

VI.1. Let us recall that a variety $\mathbb{V}$ of universal algebras is determined by a similarity type $\mathbb{T}$ (we always assume that $\mathbb{T}$ is *finitary*, i.e., $\mathbb{T} = \bigcup_{n \in \omega} \mathbb{T}_n$ where $\mathbb{T}_n$ consists of n-ary operational symbols) and a set Σ of equations, that is, a set of pairs of $\mathbb{T}$-terms (= polynomial symbols [4]). The pair $(\mathbb{T}, \Sigma)$ is called an *equational theory*. The clone $\mathrm{Clo}(\mathbb{T}, \Sigma)$ of an equational theory $(\mathbb{T}, \Sigma)$ is an ω-sorted algebra, the nth sort of which is formed by all the $\mathbb{T}$-terms over n symbols $x_1^{(n)}, \ldots, x_n^{(n)}$ with the equality derived from the equations in Σ, the symbols $x_1^{(n)}, \ldots, x_n^{(n)}$ are constants (= nullary operations) of the nth sort of $\mathrm{Clo}(\mathbb{T}, \Sigma)$ and the operations S_n^m of the clone are just substitutions. For more detailed descriptions and more complete references, see [13].

VI.2. As in [13], we write $\mathrm{Clo}(X) \leq \mathrm{Clo}(Y)$ (regardless of whether X or Y is a space or an equational theory; for an abstract theory of clones see [9, 10]) whenever there is a clone homomorphism $h: \mathrm{Clo}(X) \to \mathrm{Clo}(Y)$.

Let $\mathbb{V}$ be the variety determined by $(\mathbb{T}, \Sigma)$, let $F_{\mathbb{V}}(P)$ denote the free algebra of $\mathbb{V}$ over a set P. By [12], every Tichonov topology t on P can be extended to a topology $t_{\mathbb{V}}$ on $F_{\mathbb{V}}(P)$ such that $(F_{\mathbb{V}}(P), t_{\mathbb{V}})$ is a free object over the space

(P, t) in the category of topological $\mathbb{V}$-algebras. This implies that $\mathrm{Clo}(\mathbb{T}, \Sigma) \leq \mathrm{Clo}(F_{\mathbb{V}}(P), t_{\mathbb{V}})$. In [13], it is conjectured that the reverse inequality could be true for $(\mathbb{T}, \Sigma)$ of finite type (with $\mathbb{T}_0 \neq \varnothing$) and a Cook continuum (P, t), see [3].

Here we investigate only *free equational theories,* that is, those $(\mathbb{T}, \Sigma)$ with $\Sigma = \varnothing$. For these theories, we show that

(a) $\mathrm{Clo}(\mathbb{T}, \varnothing) \geq \mathrm{Clo}(F_{\mathbb{V}}(P), t_{\mathbb{V}})$ for no space (P, t) whatsoever,

(b) however, there exist many metrics τ on the initial $\mathbb{V}$-algebra $F_{\mathbb{V}}(\varnothing)$ such that the clone of each space $(F_{\mathbb{V}}(\varnothing), \tau)$ is isomorphic to $\mathrm{Clo}(\mathbb{T}, \varnothing)$ provided $\mathbb{T}$ has sufficiently many constants (which means that $\mathrm{card}\, \mathbb{T}_0 \geq 2^{\aleph_0} + \mathrm{card} \bigcup_{n=1}^{\infty} \mathbb{T}_n$).

As a corollary of (b), we conclude that, for an arbitrary equational theory $(\mathbb{T}, \Sigma)$,

> $\mathrm{Clo}(\mathbb{T}, \Sigma)$ is a homomorphic image of $\mathrm{Clo}(X)$ for a suitable (metrizable) space X iff $\mathbb{T}_0 \neq \varnothing$.

VI.3. The statement (a) in VI.2 will be demonstrated in its simplest case, namely that of $\mathbb{T} = \mathbb{T}_0 \cup \mathbb{T}_1$, $\mathbb{T}_0 = \{c\}$, $\mathbb{T}_1 = \{\sigma\}$. A generalization to every free equational theory is straightforward.

For any set P (including $P = \varnothing$), the underlying set of $F_{\mathbb{V}}(P)$ is a disjoint union of ω copies of $Q = P \cup \{c\}$, where $c \notin P$ (the nth copy is just $\{\sigma^n(x) | x \in Q\}$). Given a topology t on P, take the topology u on Q such that (Q, u) is a coproduct (in Top) of the space (P, t) and the one-point space $\{c\}$. Then the topology $t_{\mathbb{V}}$ on $F_{\mathbb{V}}(P)$ is such that $X = (F_{\mathbb{V}}(P), t_{\mathbb{V}})$ is a coproduct (in Top) of ω copies of (Q, u). We intend to prove that $\mathrm{Clo}(X)$ admits no homomorphism into the clone of $(\mathbb{T}, \varnothing)$.

Suppose that $h = \{h_k | k \in \omega\}$ is such a homomorphism.

(α) In the notation of the previous parts, h_1 must send the identity map $\pi_1^{(1)} : X \to X$ to $x_1^{(1)}$. We show that h_1 sends also any constant map $X \to X$ to $x_1^{(1)}$.

Since X is a coproduct (in Top) of ω copies of (Q, u), we can express its underlying set as $\bigcup_{n \in \omega} \{n\} \times Q$ and, for every map $m \colon \omega \to \omega$, the map $\tilde{m} : X \to X$, defined by $\tilde{m}(n, x) = (m(n), x)$, is continuous. We define $m_1, m_2 : \omega \to \omega$ by

$$m_1(n) = 2n \quad \text{and} \quad m_2(n) = 2n + 1 \qquad \text{for all } n \in \omega.$$

Clearly, there exist continuous $r_1, r_2 : X \to X$ such that both maps $r_1 \circ \tilde{m}_1$ and $r_2 \circ \tilde{m}_2$ coincide with the identity $\pi_1^{(1)}$.

Let a be an arbitrary point of X. Let $p_1, p_2 : X \to X$ be the continuous maps given by

$$\begin{array}{lll} \text{for } n \text{ even,} & p_1(n, x) = (n, x) \ \text{ and } \ p_2(n, x) = a; \\ \text{for } n \text{ odd,} & p_1(n, x) = a \ \text{ and } \ p_2(n, x) = (n, x). \end{array}$$

Clearly, $p_1 \circ \tilde{m}_1 = \tilde{m}_1$, $p_2 \circ \tilde{m}_2 = \tilde{m}_2$, and $p_1 \circ p_2$ is the constant map to a. Rewriting this in terms of S_1^1 in $\mathrm{Clo}(X)$, we get $S_1^1(r_i, \tilde{m}_i) = \pi_1^{(1)}$, $S_1^1(p_i, \tilde{m}_i) = \tilde{m}_i$, $i = 1, 2$.

(β) The constant $x_1^{(1)}$ and the operation S_1^1 of the first sort of the clone of $(\mathbb{T}, \varnothing)$ satisfy the implication

$$S_1^1(s_1, s_2) = x_1^{(1)} \Rightarrow s_1 = x_1^{(1)} = s_2.$$

Hence, using $h_1(\pi_1^{(1)}) = x_1^{(1)}$, we get $h_1(r_i) = h_1(\tilde{m}_i) = x_1^{(1)}$, $i = 1, 2$. Consequently $h_1(p_i) = x_1^{(1)}$, $i = 1, 2$, so that h_1 sends also the constant $S_1^1(p_1, p_2)$ to $x_1^{(1)}$.

(γ) Let $\pi_1^{(2)}, \pi_2^{(2)} : X \times X \to X$ be the projections. Then h_2 sends them to $x_1^{(2)}$ and $x_2^{(2)}$ with $x_1^{(2)} \neq x_2^{(2)}$. Choose a point $a \in X$ and denote by $\tilde{a} : X \to X$ the constant map to a. We have $\tilde{a}\circ\pi_1^{(2)} = \tilde{a}\circ\pi_2^{(2)}$, i.e., $S_2^1(\tilde{a}, \pi_1^{(2)}) = S_2^1(\tilde{a}, \pi_2^{(2)})$ in $\mathrm{Clo}(X)$. Hence necessarily $S_2^1(h_1(\tilde{a}), h_2(\pi_1^{(2)})) = S_2^1(h_1(\tilde{a}), h_2(\pi_2^{(2)}))$ in the clone of $(\mathbb{T}, \varnothing)$, so that $x_1^{(2)} = S_2^1(x_1^{(1)}, x_1^{(2)}) = S_2^1(x_1^{(1)}, x_2^{(2)}) = x_2^{(2)}$ which is a contradiction.

VI.4. Let $(\mathbb{T}, \varnothing)$ be an equational theory with sufficiently many constants, and let $\mathbb{V}$ be the variety it determines. In VI.4–5 below, we construct a metric τ on $F_{\mathbb{V}}(\varnothing)$ such that $(F_{\mathbb{V}}(\varnothing), \tau)$ fulfils (b) in VI.3.

Let $X = (P, \varrho)$ be an extremally B-semirigid space such that $\operatorname{card} B = \operatorname{card} \mathbb{T}_0$ and $G_0 = P \setminus B$ is precisely $\mathbb{T}_0$. Denote $\mathbb{Q} = \bigcup_{n=1}^{\infty} \mathbb{T}_n$, and write B as the union of pairwise disjoint system $\{B_{q,i} | q \in \mathbb{Q}, i = 1, 2, \dots\}$ of its subsets such that $\operatorname{card} B_{q,i} = \operatorname{card} B$ for all $q \in \mathbb{Q}$, $i = 1, 2, \dots$. We set

$$B_i = \bigcup_{q\in\mathbb{Q}} B_{q,i}, \qquad G_n = G_0 \cup \bigcup_{i=1}^{n} B_i$$

and, for every $q \in \mathbb{T}_n$, we find a bijection g_q of P^n onto $B_q = \bigcup_{i=1}^{\infty} B_{q,i}$ such that g_q maps G_0^n onto $B_{q,1}$, and g_q maps $G_m^n \setminus G_{m-1}^n$ onto $B_{q,m+1}$. Clearly, $(P, \{g_q | q \in \mathbb{Q}\})$ is a free algebra of the type $\mathbb{Q}$ over $\mathbb{T}_0$, that is, an initial algebra of the type $\mathbb{T}$.

Now, we construct a metric τ on P analogously to III.6, as follows:

$$\begin{aligned}
\tau_0 &= \varrho ; \\
\tau_\alpha &= \inf_{\beta<\alpha} \tau_\beta \quad \text{for limit ordinals } \alpha; \\
\tau_\alpha &= \varrho * u_\beta \quad \text{for } \alpha = \beta + 1
\end{aligned}$$

where u_β is the metric on B such that

(α) for every $q \in \mathbb{T}_n$, g_q is an isometry of (P^n, τ_β^n) onto $(B_q, u_\beta | B_q \times B_q)$;

(β) $u_\beta(x, y) = 1$ whenever $x \in B_q$, $y \in B_{q'}$, $q \neq q'$.

As in III.7, the descending chain of pseudometrics stops, the resulting pseudometric τ is a metric and the metric space $X = (P, \tau)$ has the clone isomorphic to $\mathrm{Clo}(\mathbb{T}, \varnothing)$. We sketch the proof of the last statement in VI.5 below.

VI.5. We show that, for every $n = 1, 2, \dots$, there is a one-to-one correspondence between continuous maps $X^n \to X$ and finite trees with leaves labeled by the points of G_0 $(= \mathbb{T}_0)$ or the projections $\pi_i^{(n)}: X^n \to X$, $i = 1, \dots, n$, and nodes labeled by elements of $\mathbb{Q}$ such that any node labeled by $q \in \mathbb{T}_k$ has precisely k immediate successors (for a more precise description of such trees, see, e.g., [1]).

Given such a labeled tree (T, l), there is an algorithm $\mathscr{A}_n$, analogous to that of IV.3, that gives us a continuous $f: X^n \to X$ (in IV, we had only one binary operation $g: P^2 \to P$, and this was also the reason why the trees in IV were binary; each node was, in fact, labeled by g). Conversely, for a given continuous map $f\colon X^n \to X$, we have to describe the procedure $\mathscr{A}_n^{-1}$. By II.5, either f is a projection $\pi_i^{(n)}$ (then $\mathscr{A}_n^{-1}(f)$ is the root labeled by $\pi_i^{(n)}$) or a constant with the value $x \in G_0$ (then $\mathscr{A}_n^{-1}(f)$ is the root labeled by x), or $f(X^n) \subseteq B$. Since X is connected, by II.1, $f(X^n)$ is also connected, and hence it is contained in a component of the space $(B, \tau|B \times B)$. But the definition of τ_α's implies that $\{B_q | q \in \mathbb{Q}\}$ is precisely the system of all its components. Hence there exists exactly one $q \in \mathbb{Q}$, say $q \in \mathbb{T}_k$, such that $f(X^n) \subseteq B_q$. We label the root by q and investigate the k maps $\pi_1^{(k)} \circ g_q^{-1} \circ f, \dots, \pi_k^{(k)} \circ g_q^{-1} \circ f\colon X^n \to X$.

The next step in the procedure $\mathscr{A}_n^{-1}$ is evident. The proof that the procedure $\mathscr{A}_n^{-1}$ stops after a finite number of steps is analogous to that in IV.4. We mention explicitly that also $\mathscr{A}_0$ gives a bijection of the set of all trees with leaves labeled by the points of G_0 onto the set P, i.e., the algorithm works also for the zeroth sort. We claim that $\mathscr{A} = \{\mathscr{A}_n | n \in \omega\}$ determines the isomorphism of the clone of $(\mathbb{T}, \varnothing)$ onto the clone of $X = (P, \tau)$. In fact, the $\mathbb{T}$-terms can be also regarded as trees so that the nth sort of the clone of $(\mathbb{T}, \varnothing)$ consists of all finite trees with leaves labeled by $x_1^{(n)}, \dots, x_n^{(n)}$ and by the elements of $\mathbb{T}_0$ and nodes labeled by elements of $\mathbb{Q} = \bigcup_{k=1}^{\infty} \mathbb{T}_k$ and every node labeled by a $q \in \mathbb{T}_k$ has precisely k immediate successors. In any such tree t, interpreting $x_i^{(n)}$ as the projection $\pi_i^{(n)}\colon X^n \to X$ and every $q \in \mathbb{T}_0$ as a constant map $X^n \to X$ with value q, $\mathscr{A}_n$ gives us a continuous map $\mathscr{A}_n(t)\colon X^n \to X$; hence (in our interpretation) $\mathscr{A}_n$ sends the constants $x_1^{(n)}, \dots, x_n^{(n)}$ of the nth sort of the clone of $(\mathbb{T}, \varnothing)$ to the corresponding constants $\pi_1^{(n)}, \dots, \pi_n^{(n)}$ of the nth sort of $\mathrm{Clo}(X)$. The clone operations S_m^n act in the clone of $(\mathbb{T}, \varnothing)$ as follows: $S_m^n(t, s_1, \dots, s_n)$ is the tree obtained from the tree t (in variables $x_1^{(n)}, \dots, x_n^{(n)}$) so that any occurrence of the variable $x_i^{(n)}$ is replaced by the tree s_i (in the variables $x_1^{(m)}, \dots, x_m^{(m)}$). Let $\mathscr{A}_n$ send the tree t to a continuous map $g\colon X^n \to X$: that is, each label $x_i^{(n)}$ is replaced by $\pi_i^{(n)}\colon X^n \to X$ and each label $q \in \mathbb{T}_0$ by a constant map $X^n \to X$ with the value $q \in G_0$ $(= \mathbb{T}_0)$ and these maps are processed as prescribed by the tree, so that the resulting map is g. If, however, in another interpretation, we replace each label $x_i^{(n)}$ by a continuous map $h_i : X^m \to X$ and each label $q \in \mathbb{T}_0$ by a constant map $X^m \to X$ with the value $q \in G_0$ and process **these** maps as prescribed by the tree t, then the resulting map is $X^m \xrightarrow{h_1 \dot{\times} \cdots \dot{\times} h_n} X^n \xrightarrow{g} X$, which is precisely $S_m^n(g, h_1, \dots, h_n)$ (we denote the clone operations in $\mathrm{Clo}(X)$ by S_m^n as well). If $h_i\colon X^m \to X$ is the map $\mathscr{A}_m(s_i)$, then $S_m^n(g, h_1, \dots, h_n)$ is, indeed, just the map corresponding to the tree $S_m^n(t, s_1, \dots, s_n)$.

VI.6. In VI.2(b), we claim that there are *many* metrics τ on the initial algebra $F_{\mathbb{V}}(\varnothing)$ of the free equational theory $(\mathbb{T}, \varnothing)$ with sufficiently many constants such that $\mathrm{Clo}(F_{\mathbb{V}}(\varnothing), \tau)$ is isomorphic to $\mathrm{Clo}(\mathbb{T}, \varnothing)$. How many? The statement below answers this question.

Let $\mathbb{V}$ be the variety of a free equational theory $(\mathbb{T}, \varnothing)$ with $\alpha = \mathrm{card}\,\mathbb{T}_0 \geq 2^{\aleph_0} + \mathrm{card} \bigcup_{n=1}^{\infty} \mathbb{T}_n$. Let P be the underlying set of the initial algebra $F_{\mathbb{V}}(\varnothing)$ of $\mathbb{V}$. Then there exists a set $\mathbb{M}$ of metrics on P such that

(i) $\mathrm{card}\,\mathbb{M} = 2^\alpha$, and every continuous map $f\colon (P, \tau) \to (P, \tau')$ is constant whenever $\tau, \tau' \in \mathbb{M}$, $\tau \neq \tau'$,

(ii) for every $\tau \in \mathbb{M}$, $\mathrm{Clo}(P, \tau)$ is isomorphic to $\mathrm{Clo}(\mathbb{T}, \varnothing)$;
(iii) for every $\tau \in \mathbb{M}$, every n-ary operation of the algebra $F_{\mathbf{V}}(\varnothing)$ is an isometry of (P^n, τ^n) onto its image in (P, τ) and every continuous map $(P^k, \tau^k) \to (P, \tau)$ is nonexpanding.

To prove this statement, we begin with a set $\mathbb{R}$ of metrics on P mentioned in VII.14: $\operatorname{card}\mathbb{R} = 2^\alpha$ and if $\varrho, \varrho' \in \mathbb{R}$, $\varrho \neq \varrho'$, then every continuous map $f : (P, \varrho) \to (P, \varrho')$ is constant; moreover, for every $\varrho \in \mathbb{R}$, the space (P, ϱ) is extremally B-semirigid with B equal to the union of all $g_q(P^k)$ where $\{g_q | q \in \mathbb{Q}\}$ are the operations of $F_{\mathbf{V}}(\varnothing)$ of positive arity. Then, for every $\varrho \in \mathbb{R}$, we construct τ_ϱ precisely as in VI.4. Then (P, τ_ϱ) fulfils (ii) and (iii), by VI.4–5. If $\varrho, \varrho' \in \mathbb{M}$, $\varrho \neq \varrho'$ and $f : (P, \tau_\varrho) \to (P, \tau_{\varrho'})$ is a continuous map, then either f is constant or $f(P) \subseteq B$, by VII.14 again. However, in the last case, f must be constant as well. The reasoning of VI.5 (in combination with VII.14) yields the result.

VII. Proof of the Main Theorem

VII.1. The aim of this section is to prove the Main Theorem. Given a cardinal number $\alpha \geq 2^{\aleph_0}$, we have to construct an extremally B-semirigid metric space (P, ϱ) with $B \subseteq P$, $\operatorname{card} B = \operatorname{card}(P \setminus B) = \alpha$. The construction heavily uses the existence of a Cook continuum, that is, an infinite compact metric connected space H such that for each subcontinuum H_0 of H, any continuous map $f : H_0 \to H$ is either constant or the inclusion (i.e. $f(x) = x$ for all $x \in H_0$). Such a continuum was constructed in [3], and a more detailed version of the construction can be found in [11].

VII.2. The second result which we shall use is the following graph-theoretical statement:

for every $\alpha \geq 2^{\aleph_0}$ there exists a stiff set $\mathbb{S}_\alpha$ of connected directed graphs $G = (V, R)$ with $\operatorname{card} V = \alpha$ such that $\operatorname{card}\mathbb{S}_\alpha = \alpha$,

where "stiff" means that, apart from their identities, these graphs admit no compatible maps (i.e., if $g : (V, R) \to (V', R')$ is a map such that $(v, w) \in R \Rightarrow (g(v), g(w)) \in R'$, then $(V, R) = (V', R')$ and $g(v) = v$ for all $v \in V$).

In passing, we mention a well-known and stronger result: for every infinite α there exists a stiff set of such graphs of cardinality 2^α; see, for instance, [11, 10.3, p.139 and 5.3, p.72].

VII.3. We construct the required space (P, ϱ) in VII.4–8 below and then, in VII.9–13, we prove its extremal B-semirigidity.

In VII.4–8 we work with metric spaces rather than with topological spaces. We always suppose that the diameter of any space is less than or equal to 1 and, in fact, we perform the construction within the category Metr formed by all metric spaces of diameter at most 1 and by their nonexpanding maps. "Coproduct" always means the coproduct in Metr. Also "gluing" means the quotient in Metr: if (Q, σ) is an object of Metr and $m : Q \to Q'$ is a surjective map, we endow Q' by the pseudometric σ' defined by

$$\sigma'(a, b) = \inf \sum_{i=1}^{n} \sigma(x_i, y_i)$$

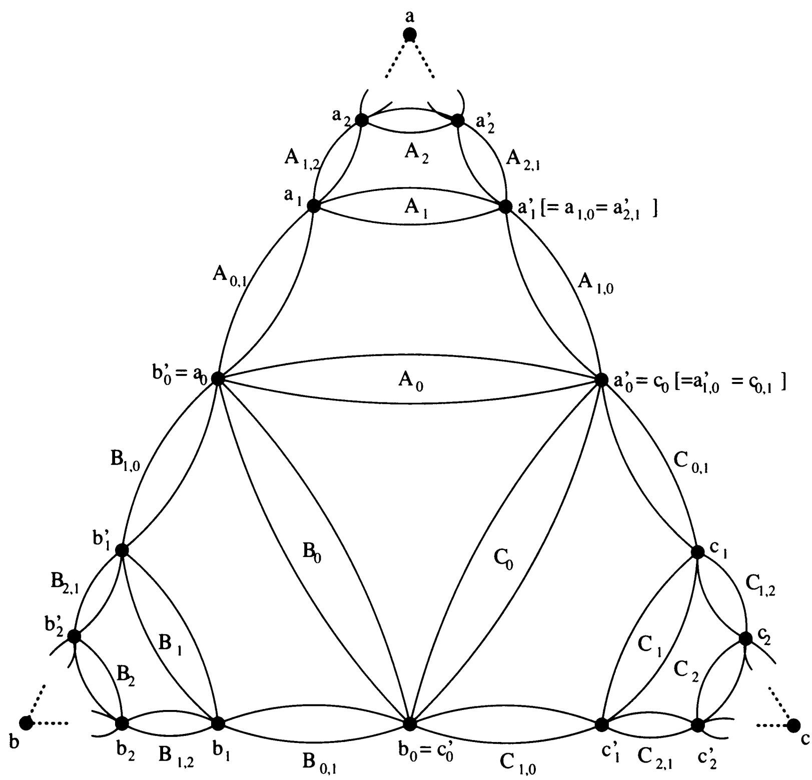

FIGURE 1. Triangle space T

where the infimum is taken over all chains $x_1 = a, y_1, x_2, y_2, \dots, x_n, y_n = b$ with $m(x_i) = m(y_{i+1})$ for $i = 1, \dots, n-1$. In our concrete situation, the gluing (i.e., the map $m : Q \to Q'$) will be so simple that it will be seen immediately that σ' is a metric.

VII.4. **The triangle space T.** We choose a collection of pairwise disjoint nondegenerate subcontinua of a Cook continuum H. They are indexed as follows:

$$\{A_n, A_{n,n+1}, A_{n+1,n}, B_n, B_{n,n+1}, B_{n+1,n}, C_n, C_{n,n+1}, C_{n+1,n} | n \in \omega\}.$$

We may suppose (by the multiplying of their metric inherited from H by suitable real numbers) that the diameter of A_n, B_n, C_n is $2^{-(n+1)}$ and of $A_{n,n+1}$, $A_{n+1,n}$, $B_{n,n+1}$, $B_{n+1,n}$, $C_{n,n+1}$, $C_{n+1,n}$ is $2^{-(n+2)}$. In each continuum, we choose two points whose distance equals the diameter, and denote them as a_n and a'_n in A_n, as $a_{n,n+1}$ and $a'_{n,n+1}$ in $A_{n,n+1}$, etc. We now glue the coproduct of all these spaces to a connected metric space as indicated by Figure 1.

Finally, we form the metric completion T of the resulting metric space by adding the points a, b, c as indicated. The space T will be called a *triangle space*. This construction is described in [11, pp. 222–224] and, in the same notation as here, also in [14]. Clearly, $\operatorname{diam} T = 1$. Let S denote the subset of the triangle space T consisting of a, b, c and of all the points in which the spaces $A_n, \dots$ are glued together, and call S the *skeleton* of T. Let $\mathbb{X}$

denote the system of the above spaces $A_n, \dots$ considered as subspaces of T (i.e., homeomorphic to the subcontinua of H); we call them *building blocks of* T. The gluing used is such that T satisfies the following statement:

(*) For every $s \in S$ and every pair of building blocks X, Y there are building blocks $X_0, \dots, X_{k+1}$ such that $X = X_0$, $Y = X_{k+1}$ $s \notin X_j$ for $j = 1, \dots, k$ and $X_j \cap X_{j+1} \neq \varnothing$ for $j = 0, \dots, k+1$.

VII.5. We shall need two triangle spaces T^1 and T^2, constructed as in VII.4. For both T^1 and T^2, we start from a collection of pairwise disjoint nondegenerate subcontinua of H; moreover we suppose that every subcontinuum used in the construction of T^1 is disjoint with any subcontinuum used in the construction of T^2. Under these assumptions,

(**) if $i, j \in \{1, 2\}$, X is a building block of T^i, Y is a building block of T^j, H_0 is a subcontinuum of X, and $f: H_0 \to Y$ is a continuous map, then either f is constant or $i = j$, $X = Y$ and f is the inclusion map.

The notation for building blocks, points, skeleton, etc. of T^1 will be that of VII.4, while building blocks, points, skeleton of T^2, will be indicated by adding of upper index 2, e.g., A_n^2, $A_{n,n+1}^2$, a^2, b^2, c^2, S^2, etc.

VII.6. **The arrow construction.** Let $G = (V, R)$ be a connected directed graph. We replace each its arrow $r = (v, w) \in R$ by a copy T_r of T^1 (we add to every point, set, ... of this copy the index r, e.g., $a_r, b_r, c_r, \dots$) as follows: in the coproduct of $\{T_r | r \in R\}$, we glue together all points c_r with $r \in R$ and denote the resulting point d_G. Moreover, we glue together

a_r and $a_{r'}$ whenever $r = (v, w_1)$ and $r' = (v, w_2)$,
b_r and $b_{r'}$ whenever $r = (v_1, w)$ and $r' = (v_2, w)$,
a_r and $b_{r'}$ whenever $r = (v, w_1)$ and $r' = (w_2, v)$.

Let $\mathscr{M}(G)$ denote the resulting space (in [11, pp. 214–215], the above construction is presented in a more general setting). We may regard the set V of vertices of G as a 1-discrete subset of $\mathscr{M}(G)$ (it is just the set where a_r, b_r are glued together). Let $e_r: T^1 \to \mathscr{M}(G)$ denote the map that sends T^1 onto its copy T_r (regarded as a subspace of $\mathscr{M}(G)$), and let $e_r(x) = x_r$ for all $x \in T^1$. The skeleton S_G of $\mathscr{M}(G)$ is $\bigcup_{r \in R} e_r(S)$. Clearly, $d_g \in S_G$ and $V \subseteq S_G$.

VII.7. **The construction of D_G.** Let $G = (V, R)$ be a connected directed graph. Choose its vertex $v \in V$. We form the space $\mathscr{M}(G)$ as in VII.6. The space D_G is formed precisely as T^2, beginning with the same collection of pairwise disjoint nondegenerate subcontinua of H with only one exception: a single subcontinuum, say A_1^2, is replaced by $\mathscr{M}(G)$. In more detail, we replace A_1^2 and its gluing points a_1^2 and $a_1'^2$ by $\mathscr{M}(G)$ with its metric multiplied by 2^{-2} (because $\operatorname{diam} \mathscr{M}(G) = 1$ and $\operatorname{diam} A_1^2 = 2^{-2}$) and its points v and d_G. (Although the constructed space D_G depends also on the choice of the vertex v, our notation does not indicate it.) When viewed as points of D_G, the points a^2, b^2, c^2 of T^2 will be denoted by a_G, b_G, c_G. For the sake of brevity, we shall suppose that $\mathscr{M}(G) \subseteq D_G$ and that the map e_r defined in VII.6 by $e_r(x) = x_r$, maps T^1 (or any of its building blocks) into D_G.

The *skeleton of* D_G is the union of the skeleton S^2 of T^2 and the skeleton S_G of $\mathscr{M}(G)$, the *building blocks of* D_G are all the building blocks of T^2 except A_1^2 (we call them *big building blocks*) and all the building blocks of T^1 (we call them *small building blocks*). Every big building block appears in D_G only once, small building blocks appear in D_G $(\operatorname{card} R)$-times. For every small building block X, $e_r : X \to D_G$ denotes the embedding of X onto its copy in $T_r \subseteq \mathscr{M}(G) \subseteq D_G$ (hence $e_r(x) = x_r$ for all $x \in X$).

VII.8. **The construction of** (P, ϱ). Let $\mathbb{S}_\alpha$ be a stiff collection of connected directed graphs $G = (V, R)$ with $\operatorname{card} V = \alpha$ and $\operatorname{card} \mathbb{S}_\alpha = \alpha$, as described in VII.2. For every $G \in \mathbb{S}_\alpha$, $G = (V, R)$, choose $v \in V$ and construct D_G as in VII.7. Then (P, ϱ) is obtained from the coproduct of all D_G, $G \in \mathbb{S}_\alpha$, by merging of all points a_G into one point a and all points c_G into one point c. We suppose that $D_G \subseteq P$ for all $G \in \mathbb{S}_\alpha$ and $D_G \cap D_{G'} = \{a, c\}$ for $G \neq G'$. On the other hand, any building block X of T^2 other than A_1^2 is no more a subset of P; let $e_G: X \to P$ denote the "identity" map of X onto its copy in D_G and write $e_G(x) = x_G$ for all $x \in X$. If X is a small building block of D_G and $G = (V, R)$, $r \in R$, we denote by $e_{G,r}: X \to P$ the composite of $e_r: X \to D_G$ with the inclusion $D_G \to P$. Also, let $e_{G,r}: T^1 \to P$ denote the maps equal to $e_{G,r}: X \to P$ for any building block of T^1. The *skeleton* of (P, ϱ) is defined as the union of all skeletons of all D_G, $G \in \mathbb{S}_\alpha$, and we denote it by $\operatorname{sk} P$. We prove that (P, ϱ) is extremally B-semirigid, where B is the subset $\{b_G | G \in \mathbb{S}_\alpha\}$. Clearly, B is 1-discrete in (P, ϱ). Since $\alpha \geq 2^{\aleph_0}$, we have $\operatorname{card} B = \operatorname{card}(P \setminus B) = \alpha$.

VII.9. Let t be a Hausdorff topology on P coarser than the topology determined by ϱ and coinciding with it on $P \setminus B$, which is open in both. To finish the proof of the Main Theorem, it suffices to prove that (P, t) is B-semirigid. First, observe that for every building block X of T^1 and every building block Y of T^2 other than A_1^2, the sets $e_{G,r}(X)$ and $e_G(Y)$ are compact in $P \setminus B$, and hence closed in the Hausdorff space (P, t), and that their topology is the same in (P, t) and in (P, ϱ). The definition of the skeleton $\operatorname{sk} P$ does not depend on topology, so that we use it for (P, t) as for (P, ϱ). Observe also that $B \subseteq \operatorname{sk} P$.

VII.10. **Lemma.** (a) *Let X be a building block of T^2, $X \neq A_1^2$. Let $f: X \to (P, t)$ be a continuous map. Then either f is constant or $f(X) \subseteq B$ or there exists (a unique) $G \in \mathbb{S}_\alpha$ such that $f = e_G$.*

(b) *Let X be a building block of T^1. Let $f: X \to (P, t)$ be a continuous map. Then either f is constant or $f(X) \subseteq B$, or there exists (a unique) $G = (V, R) \in \mathbb{S}_\alpha$ and (a unique) $r \in R$ such that $f = e_{G,r}$.*

Proof. (1) Let us suppose that $f(X) \subseteq \operatorname{sk} P$. Since components of $\operatorname{sk} P \setminus B$ consist of single points and $f(X)$ is connected, either f is constant or $f(X) \subseteq B$.

(2) Let $f(X)$ intersect $P \setminus \operatorname{sk} P$. Hence there exist $x \in X$ and Z with $f(x) \in Z \setminus \operatorname{sk} P$, where either $Z = e_G(Y)$ for a building block Y of T^2, $Y \neq A_1^2$ and some $G \in \mathbb{S}_\alpha$ or $Z = e_{G,r}(Y)$ for a building block Y of T^1 and some $G = (V, R) \in \mathbb{S}_\alpha$ and $r \in R$. Since $Z \setminus \operatorname{sk} P$ is open in $P \setminus B$, it is open in (P, t). Hence $\mathscr{O} = f^{-1}(Z \setminus \operatorname{sk} P)$ is open in X. Clearly, $x \in \mathscr{O}$.

(2,1) Let us suppose that $X \setminus \mathscr{O} \neq \varnothing$. Let C be the component of x in $\mathscr{O}$. By a well-known theorem (see, e.g., [8]), the closure of C intersects the boundary of $\mathscr{O}$ hence the closure of $f(C) \subseteq Z \setminus \operatorname{sk} P$ intersect $Z \cap \operatorname{sk} P$. Let us denote by H_0 the closure of C. Then H_0 is a subcontinuum of X and $f(H_0)$ is a nondegenerate subcontinuum of Z. Hence, by $(**)$ in VII.5, X and Z are copies of the same subcontinuum of the Cook continuum H and either $f = e_g$ for some $G \in \mathbb{S}_\alpha$ (whenever X is a building block of T^2, $X \neq A_1^2$) or $f = e_{G,r}$ for some $G = (V, R) \in \mathbb{S}_\alpha$, $r \in R$ (whenever X is a building block of T^1).

(2,2) The remaining case is that $X \setminus \mathscr{O} = \varnothing$, i.e. $f(X) \subseteq Z \setminus \operatorname{sk} P$. Then f must be constant, by $(**)$ in VII.5 again.

VII.11. **Lemma.** *Let $f : T^1 \to (P, t)$ be a continuous map. Then either f is constant or $f(T^1) \subseteq B$ or there exist $G = (V, R) \in \mathbb{S}_\alpha$ and $r \in R$ such that $f(x) = e_{G,r}(x)$ for all $x \in T^1$.*

Proof. We use VII.10 and $(*)$ in VII.4.

VII.12. **Lemma.** *Let $G = (V, R)$ be in $\mathbb{S}_\alpha$, let $\mathscr{M}(G)$ be as in* VII.6. *Let $f: \mathscr{M}(G) \to (P, t)$ be a continuous map. Then either f is constant or $f(\mathscr{M}(G)) \subseteq B$, or f is the inclusion of $\mathscr{M}(G)$ into P.*

Proof. If f maps some T_r, $r \in R$, into B, then it maps the whole $\mathscr{M}(G)$ into B. If f is constant on some T_r, $r \in R$, with the value in $P \setminus B$, then f is constant on $\mathscr{M}(G)$. Both these statements follow from VII.11 and the connectedness of the graph G. In the remaining case, f must be the inclusion of $\mathscr{M}(G)$ into $D_G \subseteq P$ because $\mathbb{S}_\alpha$ is a stiff set of connected graphs.

VII.13. **Lemma.** *Let $G = (V, R)$ be in $\mathbb{S}_\alpha$, let D_G be as in* VII.7. *Let $f : D_G \to (P, t)$ be a continuous map. Then either f is constant or $f(D_G) \subseteq B$, or f is the inclusion of D_G into P.*

Proof. By VII.12, VII.10, and $(*)$ in VII.4.

Corollary. *Let $f: (P, t) \to (P, t)$ be a continuous map. Then either f is constant or $f(P) \subseteq B$ or f is the identity; hence (P, t) is B-semirigid.*

VII.14. *Remark.* For every $\alpha \geq 2^{\aleph_0}$ there exists a set $\mathbb{M}$ of metrics on the set P such that $\operatorname{card} \mathbb{M} = 2^\alpha$, the space (P, ϱ) is extremally B-semirigid for every $\varrho \in \mathbb{M}$ and if $\varrho, \varrho' \in \mathbb{M}$, $\varrho \neq \varrho'$, then every continuous map $f: (P, \varrho) \to (P, \varrho')$ is constant (i.e., $\{(P, \varrho) | \varrho \in \mathbb{M}\}$ is a stiff set of spaces). Moreover, if t (resp t') is a Hausdorff topology on P coarser than the topology determined by ϱ (resp. ϱ') and coinciding with it on $P \setminus B$, B closed in both, then every continuous map $f: (P, t) \to (P, t')$ is either constant or maps the whole P into B.

All this follows immediately from VII.4–VII.13 if we use the fact, mentioned in VII.2, that there exists a stiff set $\mathbb{S}$ of graphs $G = (V, R)$ with $\operatorname{card} V = \alpha$ such that $\operatorname{card} \mathbb{S} = 2^\alpha$. In fact, we decompose $\mathbb{S}$ into 2^α disjoint parts, each of the cardinality α, and construct the metrics ϱ precisely as in VII.4–8 for each of these parts of $\mathbb{S}$. The proof that $\{(P, \varrho) | \varrho \in \mathbb{M}\}$ is a stiff set of spaces is contained implicitly in VII.9–13, and so is the proof of the last statement about $f: (P, t) \to (P, t')$.

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